A NUMERICAL INVESTIGATION OF THE BENDING INSTABILITY OF THIN JETS OF DROPPING LIQUIDS

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In [1, 2] a system of quasi-one-dimensional equations of thin jets of dropping liquids was derived which makes it possible, in particular, to investigate the process of growth of bending perturbations of high-velocity jets due to the action of the ambient air. These equations were solved in the limit of small perturbations, which made it possible to determine the corresponding increment in a linear approximation [3]. In the present article we give certain results of a numerical solution of the quasi-one-dimensional equations of jet dynamics [1, 2] for the case of finite plane bending perturbations of jets of Newtonian viscous liquids with a round cross section.

1. Fundamental Equations

First of all, let us discuss the variant of the problem without allowance for the air drag force. Then perturbations in the form of standing waves with an amplitude which grows with time correspond to the case of an infinite initially straight jet. For a sufficiently viscous liquid one can neglect the inertial terms in comparison with the viscous terms in all the equations of the problem except for the projection of the momentum equation onto the normal to the jet axis (in the latter equation the viscous terms are small — they have the order of the force of cutting apart). After transformations, details of which are presented in [1], we represent the quasi-one-dimensional equations of continuity, momentum (projections onto the normal and the tangent to the jet axis), and angular momentum, as well as the kinematic and geometrical equations, in the dimensionless form

$$\partial\lambda a^2/\partial t + \partial a^2 W/\partial s = 0, \tag{1.1}$$

$$\frac{\partial\lambda a^2 V_n}{\partial t} + V_\tau a^2\lambda^{-1}\frac{\partial V_n\lambda}{\partial s} + \frac{\partial a^2 V_n W}{\partial s} + a^2 W V_\tau \lambda k = \frac{1}{\mathrm{Re}}\left(\frac{\partial Q_n}{\partial s} + \lambda P k\right) - J\frac{ka^2}{\lambda},$$

$$V_\tau = C\int_0^s \frac{\lambda}{a^2}\,ds + \int_0^s kV_n\lambda ds,\quad C = -\frac{\int_0^{1/4} kV_n\lambda ds}{\int_0^{1/4}\lambda/a^2 ds},$$

$$P = 3a^2\left(\lambda^{-1}V_{\tau,s} - kV_n\right),$$

$$Q_n = -\frac{a_0^2}{4\lambda l^2}\frac{\partial}{\partial s}\left\{a^2\left[\frac{3}{\lambda}\frac{\partial}{\partial s}\left(\frac{1}{\lambda}\frac{\partial V_n}{\partial s} + kV_\tau\right) - \frac{9}{2}\frac{k}{\lambda}\frac{\partial V_\tau}{\partial s} + \frac{9}{2}k^2 V_n\right]\right\},$$

$$W = V_\tau - V_n \mathrm{H}_{,s},\ k = (\partial^2\mathrm{H}/\partial s^2)[1 + (\partial\mathrm{H}/\partial s)^2]^{-3/2},$$

$$\partial\mathrm{H}/\partial t = V_n\lambda,\ \lambda = [1 + (\partial\mathrm{H}/\partial s)^2]^{1/2},$$

where a is the radius of the jet normalized to the initial value a_0; t, time; s, a parameter measured along the axis of the unperturbed jet; V_n and V_τ, projections of the velocity at the axis of the jet onto the normal and the tangent to the axis; H, departure of the jet axis from a straight line; P and Q_n, longitudinal force and the force of cutting apart in a cross section of the jet; k, curvature of the jet axis. Moreover, as the linear and time scales we take the wavelength of the most rapidly growing small bending perturbation and the characteristic time of its growth,

$$l = 2\pi\left(\frac{9}{8}\frac{\mu^2 a_0^4}{\rho\rho_1 U_0^2}\right)^{1/6},\quad T = \left(\frac{\rho\mu a_0^2}{\rho_1^2 U_0^4}\right)^{1/3},$$

while μ/T is taken as the stress scale. Here μ is the coefficient of viscosity of the liquid; ρ and ρ_1 are the liquid and air densities; U_0 is the velocity of the unperturbed jet. As the similarity criteria we introduce the designations $\mathrm{Re} = \rho l^2/\mu T$ and $J = \rho_1 U_0^2 T^2/\rho l^2$.

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The initial perturbation was assigned in the form

$$V_n = V_\tau = 0,\ a = 1,\ \mathrm{H} = \mathrm{H}_0 \sin 2\pi s,\ \mathrm{H}_0 = (5\cdot 10^{-4} - 5\cdot 10^{-2}) \tag{1.2}$$

with the boundary conditions

$$\begin{gathered} a(s) = a(-s),\ V_n = V_\tau = \mathrm{H} = 0,\ s = 0, \\ a(1/4 + s) = a(1/4 - s),\ V_n(1/4 + s) = V_n(1/4 - s), \\ \mathrm{H}(1/4 + s) = \mathrm{H}(1/4 - s),\ V_\tau = 0,\ s = 1/4. \end{gathered} \tag{1.3}$$

In the given case it is sufficient to analyze only a quarter of the wavelength of the perturbation.

In the presence of an air drag force (taken into account using the drag of a cylinder in transverse streamline flow; see [1]) and a sufficiently high liquid viscosity the motion of the jet can be clearly separated into two components of different nature. One of them is the drift of the perturbation of the jet as a whole in the direction opposite to the motion of the jet, and the second is the deformation of the jet against the background of such drift under the action of the "lifting" component of the aerodynamic force [1, 3]. If the jet has a certain initial curvature, so that the drag force is different from zero, then, as is easy to ascertain, the drifting inertial motion remains even in the limit of an infinitely high viscosity, when bending deformations become infinitely slow and inertial effects in their development can be neglected. As a consequence of drift the jet axis can acquire a rather complicated shape and "whipouts" develop, so that the above-indicated parameterization of the jet axis must be rejected in favor of Lagrangian parameterization.

Let the initial perturbation of the jet axis be assigned by the sinusoid $\mathrm{H} = \mathrm{H}_0 \sin 2\pi\xi$, where ξ is the Cartesian coordinate measured along the axis of the unperturbed jet (the bending takes place in the $\xi\eta$ plane). Then the value of the ξ coordinate for a given liquid particle at the initial time will be taken as its Lagrangian parameter s. For such a parameterization one must take $ds/dt = 0$ in the kinematic equations of [1, 2]. Retaining the scales introduced earlier and leaving the inertial terms as before only in the projection of the momentum equation onto the normal to the jet axis, after a series of transformations (see [1]) we obtain the quasi-one-dimensional equations of the jet in the form

$$\begin{gathered} a^2 = \lambda_0/\lambda,\ \lambda = (\xi_{,s}^2 + \mathrm{H}_{,s}^2)^{1/2},\quad \lambda_0 = [1 + (2\pi\mathrm{H}_0\cos 2\pi s)^2]^{1/2}, \\ \frac{\partial V_n}{\partial t} + V_\tau\left(\frac{1}{\lambda}\frac{\partial V_n}{\partial s} + kV_\tau\right) = \frac{1}{\mathrm{Re}}\left(\frac{1}{\lambda a^2}\frac{\partial Q_n}{\partial s} + k\Sigma_{\tau\tau}\right) - JF(s), \\ \Sigma_{\tau\tau} = 3\left(\lambda^{-1}V_{\tau,s} - kV_n\right), \\ Q_n = -\frac{a_0^2}{4\lambda l^2}\frac{\partial}{\partial s}\left\{a^2\left[\frac{3}{\lambda}\frac{\partial}{\partial s}\left(\frac{1}{\lambda}\frac{\partial V_n}{\partial s} + kV_\tau\right) - \frac{9}{2}\frac{k}{\lambda}\frac{\partial V_\tau}{\partial s} + \frac{9}{2}k^2V_n\right]\right\}, \\ \frac{\partial\xi}{\partial t} = V_\tau n_\eta - V_n\tau_\eta,\quad \frac{\partial \mathrm{H}}{\partial t} = V_n\tau_\xi - V_\tau n_\xi, \\ \tau_\xi = [1 + (\mathrm{H}_{,s}/\xi_{,s})^2]^{-1/2},\ n_\xi = -(\mathrm{H}_{,s}/\xi_{,s})[1 + (\mathrm{H}_{,s}/\xi_{,s})^2]^{-1/2}, \\ \tau_\eta = -n_\xi,\ n_\eta = \tau_\xi,\quad V_\tau = V_{\tau 0} + C\int_0^s \frac{\lambda}{a^2}\,ds + \int_0^s kV_n\lambda\,ds, \\ C = -\int_0^{1/2} kV_n\lambda\,ds \Big/ \int_0^{1/2} \lambda/a^2\,ds, \\ V_{\tau 0} = \left\{\Phi - C\int_0^{1/2}\lambda_0\tau_\xi\left[\int_0^s \lambda/a^2\,ds\right]ds - \int_0^{1/2}\lambda_0\tau_\xi\left[\int_0^s kV_n\lambda\,ds\right]ds - \int_0^{1/2}\lambda_0V_nn_\xi\,ds\right\}\Big/\int_0^{1/2}\lambda_0\tau_\xi\,ds, \\ \frac{\partial\Phi}{\partial t} = -J\int_0^{1/2}\lambda_0 n_\xi F(s)\,ds,\quad t = 0,\quad \Phi = 0, \\ F(s) = \frac{\xi_{,s}^2(\mathrm{H}_{,ss}\xi_{,s} - \xi_{,ss}\mathrm{H}_{,s})}{(\xi_{,s}^2 + \mathrm{H}_{,s}^2)^{5/2}} + \frac{l}{\pi a_0}\frac{1}{a}\frac{(\mathrm{H}_{,s}/\xi_{,s})^2\operatorname{sgn}(\mathrm{H}_{,s}/\xi_{,s})}{1 + (\mathrm{H}_{,s}/\xi_{,s})^2}, \end{gathered} \tag{1.4}$$

where n_η, n_ξ, τ_η, and τ_ξ are the projections of the normal and tangent to the jet axis onto the axes of the Cartesian coordinate system $O_1\xi\eta$. It is assumed that at the initial time the

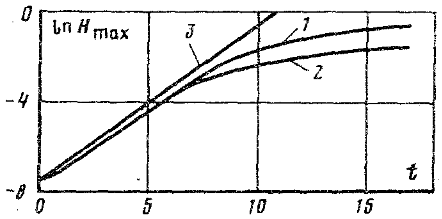

Fig. 1.

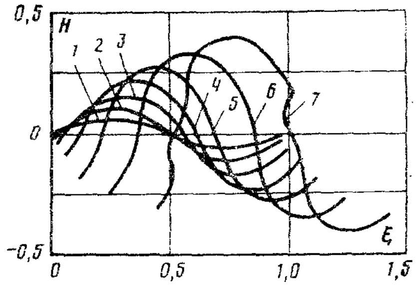

Fig. 2.

functions $\alpha = \alpha(s)$ and $V_\tau = V_\tau(s)$ have a period of 1/2, while the functions $V_n = V_n(s)$ and $H + H(s)$ satisfy the conditions

$$V_n(s) = -V_n(s + 1/2),\ H(s) = -H(s + 1/2) \tag{1.5}$$

(the latter condition, in particular, is satisfied owing to the choice of the initial perturbation of the jet axis in the form of a sinusoid). These conditions will also be satisfied for any moment of time. In the calculations it was assumed that $V_n = V_\tau = 0$ and $\alpha = 1$ at $t = 0$, a consequence of which was the initial condition for $\Phi = \int^{1/2} \lambda_0 (V_\tau \tau_\xi + V_n n_\xi)\, ds$ in (1.4).

The boundary conditions (1.3) and (1.5) used in the calculations assured the periodic extension of the solution corresponding to one perturbation wavelength to the entire jet. As a result, we investigated the so-called temporal instability of a jet in the frame of reference connected with the unperturbed jet.

We note that surface tension is unimportant in the bending of jets of highly viscous liquids, and therefore we neglected them everywhere.

2. Results of Calculations

The numerical realization of the systems of equations (1.1) and (1.4) was accomplished with an implicit finite-difference scheme, the spectrum of which well reproduced the spectrum of the linearized differential problem for small perturbations. We note that in the formal investigation of the finite-difference scheme it turns out that it has solutions which grow with time, which does not at all indicate its unsuitability but only reflects the natural instability of the physical phenomenon under investigation. Details involving the finite-difference method are presented in [1, 4].

In the calculations we investigated the development of bending perturbations of jets of very viscous Newtonian liquids ($\mu = 10\text{–}10^3$ P, $\rho = 1$ g/cm^3, $a_0 = 10^{-1}$ cm) moving in "air" ($\rho_1 = 10^{-3}$ g/cm^3) with a velocity $U_0 = 10^3$ cm/sec. The data obtained without allowance for the air drag force show that a small initial perturbation of a jet of the type (1.2) with $H_0 = 5 \cdot 10^{-4}$ rapidly becomes self-consistent and grows at the rate predicted by the linear theory of [3]. This is indicated by the comparison in Fig. 1 of the slopes of the linear sections of curves 1 ($\mu = 10$ P) and 2 ($\mu = 100$ P) with the straight line 3 corresponding to the linear theory of [3]. With a further increase in the amplitude of the perturbation its harmonic shape is distorted while the growth rate slows. The latter occurs under the action of viscous stress due to a nonlinear effect, the elongation of the jet axis during bending. Here and later the data in the figures are given in dimensionless quantities; in the case of $\mu = 10$ P the scales are $T = 0.0047$ sec and $l = 0.943$ cm, while in the case of $\mu = 100$ P they are $T = 0.01$ sec and $l = 2.02$ cm. We emphasize that data obtained for the perturbation with a wavelength corresponding to the largest increment in the linear stage of growth are presented here.

If in the calculations without allowance for the air drag force the bending perturbations consist of a system of waves standing with respect to the jet and with an amplitude which grows with time, then the presence of drag results in the perturbations being carried along the jet by the oncoming stream until they break. In Figs. 2 ($\mu = 10$ P) and 3 ($\mu = 100$ P) we present the form of a segment of the jet corresponding to one wavelength of the perturbation at different times, which are given by numbers for each of the curves. The data presented in Fig. 2 show that the rate of drift of a perturbation along the jet is about 1.5% of the velocity U_0 of the unperturbed jet. Actually, the perturbations also consist of standing

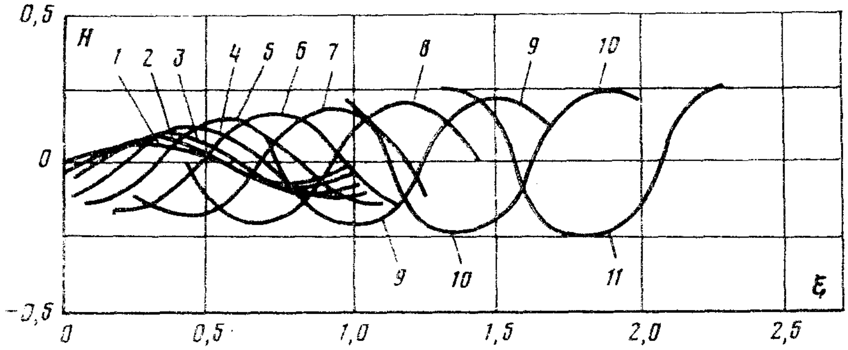

Fig. 3.

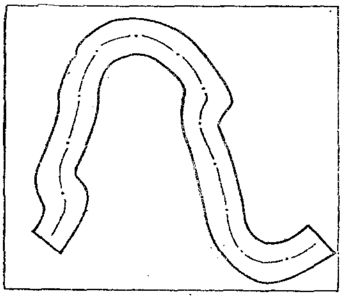

Fig. 4.

waves in this case, despite the presence of the air drag force. The jet axis very rapidly ($t = 7$) acquires the shape of a step, as a result of which a "whipout" develops. During this time the perturbation is carried about 0.47 cm by the air stream, while the jet travels 33 cm. An increase in the viscosity of the liquid with the other parameters kept unchanged results in an increase in the distance over which the perturbation wave propagates along the jet before breaking (see Fig. 3). For most of the time the shape of the perturbation before breaking depends little on the drag and is determined mainly by the "lifting" component of the aerodynamic force. This is natural, since the drag force is quadratic in the perturbation amplitude, and hence is important only for sufficiently large perturbations.

Even in late stages of deformation, when the amplitude of the perturbation approaches its wavelength, breakup of the jet does not occur. This is illustrated by Fig. 4, showing the appearance of a segment of a liquid jet with $\mu = 10$ P over one wavelength of the perturbation at the instant of breaking (it corresponds to $t = 7$ of Fig. 2). The bending is accompanied by practically synchronous thinning of the jet over its entire length. Therefore, the question of the disruption of the continuity of a jet as a result of the growth of bending perturbations remains open. The meager experimental data [5-7] show that disruption of the jet occurs after the development of bending perturbations of large enough amplitude, and the mechanism under consideration goes beyond the scope of quasi-one-dimensional processes. The disruption of jets evidently takes place up until the stabilizing influence of the viscous stresses connected with elongation of the axis during bending begins to be felt. Therefore, the results of the linear theory of [3] can be used with sufficiently good accuracy to calculate the rate of growth of bending perturbations leading to the breakup of a jet, as indicated by their comparison with the data of numerical calculations.

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NONSTATIONARY CRITICAL LAYER AND NONLINEAR INSTABILITY IN A PLANAR POISEUILLE FLOW

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One of the promising directions in the nonlinear instability theory of shear flows is related to the study of critical layers (CL) [1-6]. Stationary waves with a viscous nonlinear CL have been studied in most detail [2, 3]. Analysis of nonstationary processes of practical interest was carried out for significant simplifying restrictions [4-6]. Thus, the nonlinear development of a wave in a channel and in a boundary layer was treated only in the limiting case of a strongly nonlinear CL near a stationary one [5]. To solve the problem of generation of turbulence in these flows, however, it is necessary to have some idea of the evolution of an initially linear wave. To study nonlinear instability in a planar Poiseuille flow we use below an approach similar to that of [6] for weather instability. We consider the development of long waves, represented on the (R, α) plane by points in the neighborhood of the upper branch of the neutral curve of the linear theory (α, wave number; and R, Reynolds number). For these waves it is possible to consider independently CL and viscous regions near the channel walls. Based on analyzing a nonstationary CL, we obtain equations describing the time evolution of a wave. The transition is traced from a linear viscous CL to a wave strongly nonlinear in the increasing amplitude. As is well known, stability problems with hydrodynamic flows are largely similar in that wave—particle interactions are generated in the plasma [7-9]. In the present paper the plasma-hydrodynamic analogy provides the wave energy in a Poiseuille flow, making it possible to interpret the results obtained from the point of view of general wave theory.

1. Starting Relations. We write down the equations for a viscous incompressible fluid in the form [10]

$$\partial\zeta/\partial t + u\partial\zeta/\partial x + v\partial\zeta/\partial y = \nu\Delta\zeta; \tag{1.1}$$

$$\Delta\Psi = -\zeta, \tag{1.2}$$

where $\Delta = \partial^2/\partial x^2 + \partial^2/\partial y^2$; ζ, is the flow vorticity, Ψ is the stream function, introduced by the relations $u = \partial\Psi/\partial y$, $v = -\partial\Psi/\partial x$; and $\nu = 1/R \ll 1$ is the reciprocal Reynolds number (all the variables are assumed to have been reduced to dimensionless form). Putting

$$\Psi = \int U(y)\,dy + \psi,$$

where $U(y) > 0$ is the velocity profile in a stationary Poiseuille flow between the walls $y = 0$ and $y = 2$, we obtain the following equation for ψ:

$$\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)\Delta\psi - U''\frac{\partial\psi}{\partial x} = \frac{\partial\psi}{\partial x}\frac{\partial\Delta\psi}{\partial y} - \frac{\partial\psi}{\partial y}\frac{\partial\Delta\psi}{\partial x} + \nu\Delta^2\psi \tag{1.3}$$

(the prime denotes differentiation with respect to y). Considering a wave periodic in x, we denote the complex amplitude of the Fourier harmonic by a variable with subscript n (n = 1, 2...): $\psi_n(y, t) = \langle\psi\exp(-in\alpha\xi)\rangle$, etc., where $\xi = x - ct$, c is the phase velocity of the wave, and $\langle\ldots\rangle$ is the average over a period. In the linear approximation the profile $\psi_1(y)$ of a

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